

Classification of linearly compact simple Nambu-Poisson algebras

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Abstract

We introduce the notion of universal odd generalized Poisson superalgebra associated to an associative algebra A , by generalizing a construction made in [5]. By making use of this notion we give a complete classification of simple linearly compact (generalized) n -Nambu-Poisson algebras over an algebraically closed field of characteristic zero.

Introduction

In 1973 Y. Nambu proposed a generalization of Hamiltonian mechanics, based on the notion of n -ary bracket in place of the usual binary Poisson bracket [9]. Nambu dynamics is described by the flow, given by a system of ordinary differential equations which involves $n - 1$ Hamiltonians:

$$(0.1) \quad \frac{du}{dt} = \{u, h_1, \dots, h_{n-1}\}.$$

The (only) example, proposed by Nambu is the following n -ary bracket on the space of functions in $N \geq n$ variables:

$$(0.2) \quad \{f_1, \dots, f_n\} = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^n.$$

He pointed out that this n -ary bracket satisfies the following axioms, similar to that of a Poisson bracket:

$$(\text{Leibniz rule}) \quad \{f_1, \dots, f_i \tilde{f}_i, \dots, f_n\} = f_i \{f_1, \dots, \tilde{f}_i, \dots, f_n\} + \tilde{f}_i \{f_1, \dots, f_i, \dots, f_n\};$$

$$(\text{skewsymmetry}) \quad \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\} = (\text{sign} \sigma) \{f_1, \dots, f_n\}.$$

Twelve years later this example was rediscovered by F. T. Filippov in his theory of n -Lie algebras which is a natural generalization of ordinary (binary) Lie algebras [7]. Namely, an n -Lie algebra is a vector space with n -ary bracket $[a_1, \dots, a_n]$, which is skewsymmetric (as above) and satisfies the following Filippov-Jacobi identity:

$$(0.3) \quad [a_1, \dots, a_{n-1}, [b_1, \dots, b_n]] = [[a_1, \dots, a_{n-1}, b_1], b_2, \dots, b_n] + [b_1, [a_1, \dots, a_{n-1}, b_2], b_3, \dots, b_n] + \dots \\ + [b_1, \dots, b_{n-1}, [a_1, \dots, a_{n-1}, b_n]].$$

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In particular, Filippov proved that the Nambu bracket (0.2) satisfies the Filippov-Jacobi identity.

Following Takhtajan [10], we call an n -Nambu-Poisson algebra a unital commutative associative algebra \mathcal{N} , endowed with an n -ary bracket, satisfying the Leibniz rule, skew-symmetry and Filippov-Jacobi identity. Of course for $n = 2$ this is the definition of a Poisson algebra.

In [4] we classified simple linearly compact n -Lie algebras with $n > 2$ over a field \mathbb{F} of characteristic 0. The classification is based on a bijective correspondence between n -Lie algebras and pairs (L, μ) , where L is a \mathbb{Z} -graded Lie superalgebra of the form $L = \bigoplus_{j=-1}^{n-1} L_j$ satisfying certain additional properties, and $L_{n-1} = \mathbb{F}\mu$, thereby reducing it to the known classification of simple linearly compact Lie superalgebras and their \mathbb{Z} -gradings [8], [1]. For this construction we used the universal \mathbb{Z} -graded Lie superalgebra, associated to a vector superspace.

In the present paper we use an analogous correspondence between linearly compact n -Nambu-Poisson algebras and certain "good" pairs (\mathcal{P}, μ) , where \mathcal{P} is a \mathbb{Z}_+ -graded odd Poisson superalgebra $\mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j$ and $\mu \in \mathcal{P}_{n-1}$ is an element of parity $n \bmod 2$. For this construction we use the universal \mathbb{Z} -graded odd Poisson superalgebra, associated to an associative algebra, considered in [5]. As a result, using the classification of simple linearly compact odd Poisson superalgebras [3], we obtain the following theorem.

Theorem 0.1 *For $n > 2$, any simple linearly compact n -Nambu-Poisson algebra is isomorphic to the algebra $\mathbb{F}[[x_1, \dots, x_n]]$ with the n -ary bracket (0.2).*

Note the sharp difference with the Poisson case, when each algebra $\mathbb{F}[[p_1, \dots, p_n, q_1, \dots, q_n]]$ carries a Poisson bracket

$$(0.4) \quad \{f, g\}_P = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right),$$

making it a simple linearly compact Poisson algebra (and these are all, up to isomorphism [2]).

In the present paper we treat also the case of a generalized n -Nambu-Poisson bracket, which is an n -ary analogue of the generalized Poisson bracket, called also the Lagrange's bracket. For the latter bracket the Leibniz rule is modified by adding an extra term:

$$\{a, bc\} = \{a, b\}c + \{a, c\}b - \{a, 1\}bc.$$

In order to treat this case along similar lines, we construct the universal \mathbb{Z} -graded generalized odd Poisson superalgebra, associated to an associative algebra, which is a generalization of the construction in [5]. Our main result in this direction is the following theorem, which uses the classification of simple linearly compact odd generalized Poisson superalgebras [3].

Theorem 0.2 *For $n > 2$, any simple linearly compact generalized n -Nambu-Poisson algebra is gauge equivalent (see Remark 1.4 for the definition) either to the Nambu n -algebra from Theorem 0.1 or to the Dzhumadil'daev n -algebra [6], which is $\mathbb{F}[[x_1, \dots, x_{n-1}]]$ with the n -ary bracket*

$$(0.5) \quad \{f_1, \dots, f_n\} = \det \begin{pmatrix} f_1 & \dots & f_n \\ \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_1} \\ \dots & \dots & \dots \\ \frac{\partial f_1}{\partial x_{n-1}} & \dots & \frac{\partial f_n}{\partial x_{n-1}} \end{pmatrix}.$$

Note again the sharp difference with the generalized Poisson case, when each algebra $\mathbb{F}[[p_1, \dots, p_n, q_1, \dots, q_n, t]]$ carries a Lagrange bracket

$$(0.6) \quad \{f, g\}_L = \{f, g\}_P + (2 - E)f \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2 - E)g,$$

where $\{f, g\}_P$ is given by (0.4) and $E = \sum_{i=1}^n (p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i})$, making it a simple linearly compact generalized Poisson algebra (and those, along with (0.4), are all, up to gauge equivalence).

Throughout the paper our base field \mathbb{F} has characteristic 0 and is algebraically closed.

1 Nambu-Poisson algebras

Definition 1.1 A generalized n -Nambu-Poisson algebra is a triple $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$ such that

- (\mathcal{N}, \cdot) is a unital associative commutative algebra;
- $(\mathcal{N}, \{\cdot, \dots, \cdot\})$ is an n -Lie algebra;
- the following generalized Leibniz rule holds:

$$(1.1) \quad \{a_1, \dots, a_{n-1}, bc\} = \{a_1, \dots, a_{n-1}, b\}c + b\{a_1, \dots, a_{n-1}, c\} - \{a_1, \dots, a_{n-1}, 1\}bc.$$

If $\{a_1, \dots, a_{n-1}, 1\} = 0$, then (1.1) is the usual Leibniz rule and $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$ is called simply n -Nambu-Poisson algebra.

For $n = 2$ Definition 1.1 is the definition of a generalized Poisson algebra. Simple linearly compact generalized Poisson (super)algebras were classified in [2, Corollary 7.1].

Example 1.2 Let $\mathcal{N} = \mathbb{F}[[x_1, \dots, x_n]]$ with the usual commutative associative product and n -ary bracket defined, for $f_1, \dots, f_n \in \mathcal{N}$, by:

$$\{f_1, \dots, f_n\} = \det \begin{pmatrix} D_1(f_1) & \dots & D_1(f_n) \\ \dots & \dots & \dots \\ D_n(f_1) & \dots & D_n(f_n) \end{pmatrix}$$

where $D_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$. Then \mathcal{N} is an n -Nambu-Poisson algebra, introduced by Nambu [9], that we will call the n -Nambu algebra (cf. [9], [7], [4]).

Example 1.3 Let $\mathcal{N} = \mathbb{F}[[x_1, \dots, x_{n-1}]]$ with the usual commutative associative product and n -ary bracket defined, for $f_1, \dots, f_n \in \mathcal{N}$, by

$$\{f_1, \dots, f_n\} = \det \begin{pmatrix} f_1 & \dots & f_n \\ D_1(f_1) & \dots & D_1(f_n) \\ \dots & \dots & \dots \\ D_{n-1}(f_1) & \dots & D_{n-1}(f_n) \end{pmatrix}$$

where $D_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n-1$. Then \mathcal{N} is a generalized Nambu-Poisson algebra that we will call the n -Dzhumadil'daev algebra (cf. [6], [4]).

Remark 1.4 Let $N = (\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$ be a generalized n -Nambu-Poisson algebra. For any invertible element $\varphi \in \mathcal{N}$ define the following bracket on \mathcal{N} :

$$(1.2) \quad \{f_1, \dots, f_n\}^\varphi = \varphi^{-1} \{\varphi f_1, \dots, \varphi f_n\}.$$

Then $N^\varphi = (\mathcal{N}, \{\cdot, \dots, \cdot\}^\varphi, \cdot)$ is another generalized n -Nambu-Poisson algebra. Indeed, the skew-symmetry of the bracket is straightforward and the Filippov-Jacobi identity for the bracket $\{\cdot, \dots, \cdot\}^\varphi$ easily follows from the Filippov-Jacobi identity for the bracket $\{\cdot, \dots, \cdot\}$. Let us check that $\{\cdot, \dots, \cdot\}^\varphi$ satisfies the generalized Leibniz rule. We have:

$$\begin{aligned} \{f_1, \dots, f_{n-1}, gh\}^\varphi &= \varphi^{-1} \{\varphi f_1, \dots, \varphi f_{n-1}, \varphi gh\} = \varphi^{-1} (\{\varphi f_1, \dots, \varphi f_{n-1}, \varphi g\} h \\ &+ \varphi g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} - \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} \varphi gh) = \{f_1, \dots, f_{n-1}, g\}^\varphi h \\ &+ g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} - \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} gh = \{f_1, \dots, f_{n-1}, g\}^\varphi h \\ &+ g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} - \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} gh + g \{f_1, \dots, f_{n-1}, h\}^\varphi \\ &- \varphi^{-1} g \{\varphi f_1, \dots, \varphi f_{n-1}, \varphi h\} = \{f_1, \dots, f_{n-1}, g\}^\varphi h + g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} \\ &- \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} gh + g \{f_1, \dots, f_{n-1}, h\}^\varphi - g \{\varphi f_1, \dots, \varphi f_{n-1}, h\} \\ &- \varphi^{-1} gh \{\varphi f_1, \dots, \varphi f_{n-1}, \varphi\} + \{\varphi f_1, \dots, \varphi f_{n-1}, 1\} gh \\ &= \{f_1, \dots, f_{n-1}, g\}^\varphi h + g \{f_1, \dots, f_{n-1}, h\}^\varphi - \{f_1, \dots, f_{n-1}, 1\}^\varphi gh. \end{aligned}$$

We shall say that the generalized Nambu-Poisson algebras N and N^φ are *gauge equivalent*.

2 Odd generalized Poisson superalgebras

Definition 2.1 An odd generalized Poisson superalgebra $(\mathcal{P}, [\cdot, \cdot], \wedge)$ is a triple such that

- (\mathcal{P}, \wedge) is a unital associative commutative superalgebra with parity p ;
- $(\Pi\mathcal{P}, [\cdot, \cdot])$ is a Lie superalgebra (here $\Pi\mathcal{P}$ denotes the space \mathcal{P} with parity $\bar{p} = p + \bar{1}$);
- the following generalized odd Leibniz rule holds:

$$(2.1) \quad [a, b \wedge c] = [a, b] \wedge c + (-1)^{(p(a)+1)p(b)} b \wedge [a, c] + (-1)^{p(a)+1} D(a) \wedge b \wedge c,$$

where $D(a) = [1, a]$. If $D = 0$, then relation (2.1) becomes the odd Leibniz rule; in this case $(\mathcal{P}, [\cdot, \cdot], \wedge)$ is called an odd Poisson superalgebra (or Gerstenhaber superalgebra). Note that D is an odd derivation of the associative product and of the Lie superalgebra bracket.

Example 2.2 Consider the commutative associative superalgebra $\mathcal{O}(m, n) = \Lambda(n)[[x_1, \dots, x_m]]$, where $\Lambda(n)$ denotes the Grassmann algebra over \mathbb{F} on n anti-commuting indeterminates ξ_1, \dots, ξ_n , and the superalgebra parity is defined by $p(x_i) = \bar{0}$, $p(\xi_j) = \bar{1}$.

Set $m = n$ and define the following bracket, known as the Buttin bracket, on $\mathcal{O}(n, n)$ ($f, g \in \mathcal{O}(n, n)$):

$$(2.2) \quad [f, g]_{HO} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} \right).$$

Then $\mathcal{O}(n, n)$ with this bracket is an odd Poisson superalgebra, which we denote by $PO(n, n)$.

Example 2.3 Consider the associative superalgebra $\mathcal{O}(n, n+1)$ with even indeterminates x_1, \dots, x_n and odd indeterminates $\xi_1, \dots, \xi_n, \xi_{n+1} = \tau$. Define on $\mathcal{O}(n, n+1)$ the following bracket ($f, g \in \mathcal{O}(n, n+1)$):

$$(2.3) \quad [f, g]_{KO} = [f, g]_{HO} + (E - 2)(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} (E - 2)(g),$$

where $[\cdot, \cdot]_{HO}$ is the Buttin bracket (2.2) and $E = \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$ is the Euler operator. Then $\mathcal{O}(n, n+1)$ with bracket $[\cdot, \cdot]_{KO}$ is an odd generalized Poisson superalgebra with $D = -2 \frac{\partial}{\partial \tau}$ [1, Remark 4.1], which we denote by $PO(n, n+1)$.

Remark 2.4 Let $P = (\mathcal{P}, [\cdot, \cdot], \cdot)$ be an odd generalized Poisson superalgebra. For any invertible element $\varphi \in \mathcal{P}$, such that $p(\varphi) = \bar{0}$ and $[\varphi, \varphi] = 0$, define the following bracket on P :

$$(2.4) \quad [a, b]^\varphi = \varphi^{-1} [\varphi a, \varphi b].$$

Then $P^\varphi = (\mathcal{P}, [\cdot, \cdot]^\varphi, \cdot)$ is another odd generalized Poisson superalgebra, with derivation

$$D_\varphi(a) = [1, a]^\varphi = [\varphi, a] - D(\varphi)a.$$

The odd generalized Poisson superalgebras P and P^φ are called *gauge equivalent* (cf. [3, Example 3.4]). Note that the associative products in P and P^φ are the same.

Theorem 2.5 [3, Corollary 9.2]

- a) Any simple linearly compact odd generalized Poisson superalgebra is gauge equivalent to $PO(n, n)$ or $PO(n, n+1)$.
- b) Any simple linearly compact odd Poisson superalgebra is isomorphic to $PO(n, n)$.

Definition 2.6 A \mathbb{Z} -graded (resp. \mathbb{Z}_+ -graded) odd generalized Poisson superalgebra is an odd generalized Poisson superalgebra $(\mathcal{P}, [\cdot, \cdot], \wedge)$ such that $(\Pi\mathcal{P}, [\cdot, \cdot])$ is a \mathbb{Z} -graded Lie superalgebra: $\Pi\mathcal{P} = \bigoplus_{j \in \mathbb{Z}} \Pi\mathcal{P}_j$ (resp. a \mathbb{Z} -graded Lie superalgebra of depth 1: $\Pi\mathcal{P} = \bigoplus_{j \geq -1} \Pi\mathcal{P}_j$) and (\mathcal{P}, \wedge) is a \mathbb{Z} -graded commutative associative superalgebra: $\mathcal{P} = \bigoplus_{k \in \mathbb{Z}} \mathcal{Q}_k$ (resp. a \mathbb{Z}_+ -graded commutative associative superalgebra: $\mathcal{P} = \bigoplus_{k \in \mathbb{Z}_+} \mathcal{Q}_k$) such that $\mathcal{P}_j = \Pi\mathcal{Q}_{j+1}$.

Example 2.7 Let us consider the odd Poisson superalgebra $PO(n, n)$ (resp. $PO(n, n+1)$). Set $\deg x_i = 0$ and $\deg \xi_i = 1$ for every $i = 1, \dots, n$ (resp. $\deg x_i = 0$, $\deg \xi_i = 1$ for every $i = 1, \dots, n$ and $\deg \tau = 1$). Then $PO(n, n)$ (resp. $PO(n, n+1)$) becomes a \mathbb{Z}_+ graded odd (resp. generalized) Poisson superalgebra with

$$\mathcal{Q}_j = \{f \in \mathcal{O}(n, n) \mid \deg(f) = j\}$$

and

$$\mathcal{P}_j = \{f \in \mathcal{O}(n, n) \mid \deg(f) = j+1\}.$$

We will call this grading a grading of type $(0, \dots, 0|1, \dots, 1)$ (resp. $(0, \dots, 0|1, \dots, 1, 1)$). We thus have, for $\mathcal{P} = PO(n, n)$:

$$\Pi\mathcal{P}_{-1} = \mathcal{Q}_0 = \mathbb{F}[[x_1, \dots, x_n]]$$

and, for $j \geq 0$,

$$\Pi\mathcal{P}_j = \mathcal{Q}_{j+1} = \langle \xi_{i_1} \dots \xi_{i_{j+1}} \mid 1 \leq i_1 < \dots < i_{j+1} \leq n \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]].$$

Similarly, for $\mathcal{P} = PO(n, n+1)$, we have:

$$\mathcal{P}_{-1} = \mathcal{Q}_0 = \mathbb{F}[[x_1, \dots, x_n]]$$

$$\mathcal{P}_j = \mathcal{Q}_{j+1} = \langle \xi_{i_1} \dots \xi_{i_{j+1}} \mid 1 \leq i_1 < \dots < i_{j+1} \leq n+1 \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]].$$

Remark 2.8 From the properties of the \mathbb{Z} -gradings of the Lie superalgebras $HO(n, n)$ and $KO(n, n+1)$ (see, for example, [8]), one can deduce that the grading of type $(0, \dots, 0|1, \dots, 1)$ (resp. $(0, \dots, 0|1, \dots, 1, 1)$) is, up to isomorphisms, the only \mathbb{Z}_+ -grading of $\mathcal{P} = PO(n, n)$ (resp. $\mathcal{P} = PO(n, n+1)$) such that \mathcal{P}_{-1} is completely odd.

Remark 2.9 Let $\mathcal{P} = PO(n, n)$ or $\mathcal{P} = PO(n, n+1)$ and let \mathcal{P}^φ be an odd generalized Poisson superalgebra gauge equivalent to \mathcal{P} . Then the grading of type $(0, \dots, 0|1, \dots, 1)$ (resp. $(0, \dots, 0|1, \dots, 1, 1)$) is, up to isomorphisms, the only \mathbb{Z}_+ -grading of \mathcal{P}^φ such that \mathcal{P}_{-1}^φ is completely odd. Indeed, let $\mathcal{P}^\varphi = \oplus_{k \in \mathbb{Z}_+} \mathcal{Q}_k^\varphi = \oplus_{j \geq -1} \mathcal{P}_j^\varphi$, with $\mathcal{P}_j^\varphi = \Pi\mathcal{Q}_{j+1}^\varphi$ a \mathbb{Z}_+ -grading of \mathcal{P}^φ . Suppose that, $x_i \in \mathcal{Q}_k^\varphi$ and $\xi_i \in \mathcal{Q}_j^\varphi$ for some $1 \leq i \leq n$ and some $k, j \in \mathbb{Z}_+$. Then

$$(2.5) \quad [x_i, \xi_i]^\varphi \in \Pi\mathcal{P}_{k+j-2}^\varphi = \mathcal{Q}_{k+j-1}^\varphi.$$

On the other hand, by (2.4), we have:

$$\begin{aligned} [x_i, \xi_i]^\varphi &= \varphi^{-1}[\varphi x_i, \varphi \xi_i] = \varphi^{-1}([\varphi x_i, \varphi] \xi_i + \varphi[\varphi x_i, \xi_i] - D(\varphi x_i) \varphi \xi_i) = \\ &= [x_i, \varphi] \xi_i - D(\varphi) x_i \xi_i + [\varphi x_i, \xi_i] - D(\varphi x_i) \xi_i = \frac{\partial \varphi}{\partial \xi_i} \xi_i + \frac{1}{2} x_i \xi_i D(\varphi) - D(\varphi) x_i \xi_i + \frac{\partial \varphi}{\partial x_i} x_i + \\ &\quad + \varphi + \frac{1}{2} D(\varphi) x_i \xi_i - D(\varphi) x_i \xi_i = \frac{\partial \varphi}{\partial \xi_i} \xi_i - D(\varphi) x_i \xi_i + \frac{\partial \varphi}{\partial x_i} x_i + \varphi, \end{aligned}$$

where $D = 0$ if $\mathcal{P} = PO(n, n)$ and $D = -2\frac{\partial}{\partial \tau}$ if $\mathcal{P} = PO(n, n+1)$. Note that $[x_i, \xi_i]^\varphi$ is invertible since φ is invertible and, by (2.5), it is homogeneous, hence $k+j=1$, i.e., either $k=0$ and $j=1$ or $k=1$ and $j=0$. It follows that the only \mathbb{Z}_+ -grading of \mathcal{P}^φ such that \mathcal{P}_{-1}^φ is completely odd is the grading of type $(0, \dots, 0|1, \dots, 1)$. We can thus simply denote the graded components of \mathcal{P}^φ with respect to this grading by $\mathcal{P}_j = \Pi\mathcal{Q}_{j+1}$.

Now let $a \in \mathcal{Q}_i = \Pi\mathcal{P}_{i-1}$ and $b \in \mathcal{Q}_k = \Pi\mathcal{P}_{k-1}$. We have: $[a, b]^\varphi = [a, \varphi]b + [\varphi a, b] + (-1)^{p(a)+1}(D(\varphi)ab + D(\varphi a)b)$. Suppose that $\varphi = \sum_{j \geq 0} \varphi_j$ with $\varphi_j \in \mathcal{Q}_j$. Then one can show, using the fact that $[a, b]^\varphi \in \Pi\mathcal{P}_{i+k-2} = \mathcal{Q}_{i+k-1}$, that $[a, b]^\varphi = [a, b]^{\varphi^0}$. It follows that when dealing with the \mathbb{Z}_+ -graded odd generalized Poisson superalgebras \mathcal{P}^φ we can always assume $\varphi \in \mathcal{Q}_0$.

3 The universal odd generalized Poisson superalgebra

Definition 3.1 Let A be a unital commutative associative superalgebra with parity p . A linear map $X : A \rightarrow A$ is called a generalized derivation of A if it satisfies the generalized Leibniz rule:

$$(3.1) \quad X(bc) = X(b)c + (-1)^{p(b)p(c)} X(c)b - X(1)bc.$$

We denote by $GDer(A)$ the set of generalized derivations of A . If $X(1) = 0$, relation (3.1) becomes the usual Leibniz rule and X is called a derivation. We denote by $Der(A)$ the set of derivations of A .

Proposition 3.2 The set $GDer(A)$ is a subalgebra of the Lie superalgebra $End(A)$.

Proof. This follows by direct computations. □

Our construction of the universal odd generalized Poisson superalgebra is inspired by the one of the universal odd Poisson superalgebra explained in [5]. The universal odd Poisson superalgebra associated to A is the full prolongation of the subalgebra $Der(A)$ of the Lie superalgebra $End(A)$ (the definitions will be given below). In this section we generalize this construction when $Der(A)$ is replaced by the subalgebra $GDer(A)$.

Consider the universal Lie superalgebra $W(\Pi A)$ associated to the vector superspace ΠA : this is the \mathbb{Z}_+ -graded Lie superalgebra:

$$W(\Pi A) = \bigoplus_{k=-1}^{\infty} W_k(\Pi A)$$

where $W_{-1} = \Pi A$ and for all $k \geq 0$, $W_k(V) = \text{Hom}(S^{k+1}(\Pi A), \Pi A)$ is the vector superspace of $(k+1)$ -linear supersymmetric functions on ΠA with values in ΠA . The Lie superalgebra structure on $W(\Pi A)$ is defined as follows: for $X \in W_p(\Pi A)$ and $Y \in W_q(\Pi A)$ with $p, q \geq -1$, we define $X \square Y \in W_{p+q}(\Pi A)$ by:

$$(3.2) \quad X \square Y(a_0, \dots, a_{p+q}) = \sum_{\substack{i_0 < \dots < i_q \\ i_{q+1} < \dots < i_{q+p}}} \epsilon_a(i_0, \dots, i_{p+q}) X(Y(a_{i_0}, \dots, a_{i_q}), a_{i_{q+1}}, \dots, a_{i_{q+p}}).$$

Here $\epsilon_a(i_0, \dots, i_{p+q}) = (-1)^N$ where N is the number of interchanges of indices of odd a_i 's in the permutation $\sigma(s) = i_s$, $s = 0, \dots, p+q$. Then the bracket on $W(\Pi A)$ is given by:

$$[X, Y] = X \square Y - (-1)^{\bar{p}(X)\bar{p}(Y)} Y \square X.$$

As $GDer(A)$ is a subalgebra of the Lie superalgebra $W_0(\Pi A) = \text{End}(\Pi A)$, we can consider its full prolongation $\mathcal{G}W^{as}(\Pi A)$: this is the \mathbb{Z}_+ -graded subalgebra $\mathcal{G}W^{as}(\Pi A) = \bigoplus_{k=-1}^{\infty} \mathcal{G}W_k^{as}(\Pi A)$ of the Lie superalgebra $W(\Pi A)$ defined by setting $\mathcal{G}W_{-1}^{as}(\Pi A) = \Pi A$, $\mathcal{G}W_0^{as}(\Pi A) = GDer(\Pi A)$, and inductively for $k \geq 1$,

$$\mathcal{G}W_k^{as}(\Pi A) = \{X \in W_k(\Pi A) | [X, W_{-1}(\Pi A)] \subset \mathcal{G}W_{k-1}^{as}(\Pi A)\}.$$

Proposition 3.3 For $k \geq 0$, the superspace $\mathcal{GW}_k^{as}(\Pi A)$ consists of linear maps $X : S^{k+1}(\Pi A) \rightarrow \Pi A$ satisfying the following generalized Leibniz rule:

$$(3.3) \quad X(a_0, \dots, a_{k-1}, bc) = X(a_0, \dots, a_{k-1}, b)c + (-1)^{p(b)p(c)}X(a_0, \dots, a_{k-1}, c)b - X(a_0, \dots, a_{k-1}, 1)bc$$

for $a_0, \dots, a_{k-1}, b, c \in \Pi A$.

Proof. According to formula (3.2), for all $X \in W_p(\Pi A)$ and $Y \in W_{-1}(\Pi A) = \Pi A$, we have:

$$(3.4) \quad [X, Y](a_1, \dots, a_p) = X(Y, a_1, \dots, a_p)$$

with $a_1, \dots, a_p \in \Pi A$. Now we proceed by induction on $k \geq 0$: for $k = 0$, $\mathcal{GW}_0^{as}(\Pi A) = GDer(A)$ and equality (3.3) holds by definition of generalized derivation. Assume property (3.3) for elements in $\mathcal{GW}_{k-1}^{as}(\Pi A)$, and let X in $\mathcal{GW}_k^{as}(\Pi A)$. For any $a_0, a_1, \dots, a_{k-1}, b, c \in \Pi A$, we have by (3.4):

$$X(a_0, a_1, \dots, a_{k-1}, bc) = [X, a_0](a_1, \dots, a_{k-1}, bc).$$

By definition of $\mathcal{GW}^{as}(\Pi A)$, we have $[X, a_0] \in \mathcal{GW}_{k-1}^{as}(\Pi A)$. Using the inductive hypothesis on $[X, a_0]$, we get:

$$\begin{aligned} [X, a_0](a_1, \dots, a_{k-1}, bc) &= [X, a_0](a_1, \dots, a_{k-1}, b)c + (-1)^{p(b)p(c)}[X, a_0](a_1, \dots, a_{k-1}, c)b \\ &\quad - [X, a_0](a_1, \dots, a_{k-1}, 1)bc \end{aligned}$$

which is exactly formula (3.3) for X . □

For $X \in \Pi W_{h-1}(\Pi A)$ and $Y \in \Pi W_{k-1}(\Pi A)$ with $h, k \geq 0$, we define their concatenation product $X \wedge Y \in \Pi W_{h+k-1}(\Pi A)$ by

$$(3.5) \quad \begin{aligned} X \wedge Y(a_1, \dots, a_{h+k}) &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{h+k}}} \epsilon_a(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\ &\quad \times X(a_{i_1}, \dots, a_{i_h}) Y(a_{i_{h+1}}, \dots, a_{i_{h+k}}) \end{aligned}$$

where ϵ_a is defined as in (3.2) with $a_1, \dots, a_{h+k} \in \Pi A$.

Proposition 3.4 $(\Pi \mathcal{GW}^{as}(\Pi A), [\cdot, \cdot], \wedge)$ is a \mathbb{Z}_+ -graded odd generalized Poisson superalgebra.

We will denote $(\Pi \mathcal{GW}^{as}(\Pi A), [\cdot, \cdot], \wedge)$ by $\mathcal{G}(A)$ and call it the universal odd generalized Poisson superalgebra associated to A . The rest of this section is devoted to the proof of Proposition 3.4.

Lemma 3.5 $(\Pi \mathcal{GW}^{as}(\Pi A), \wedge)$ is a unital \mathbb{Z}_+ -graded associative commutative superalgebra with parity p .

Proof. It is already proved in [5] that $(\Pi W(\Pi A), \wedge)$ is a unital \mathbb{Z}_+ -graded associative commutative superalgebra with parity p , therefore we only need to prove that for $X \in \Pi \mathcal{GW}_{h-1}^{as}(\Pi A)$ and

$Y \in \Pi\mathcal{GW}_{k-1}^{as}(\Pi A)$ with $h, k \geq 0$, $X \wedge Y \in \Pi W_{h+k-1}(\Pi A)$ satisfies the generalized Leibniz rule (3.3). We have:

$$\begin{aligned}
& X \wedge Y(a_1, \dots, a_{h+k-1}, bc) = \\
& = \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{h+k} = h+k}} \epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\
& \quad \times X(a_{i_1}, \dots, a_{i_h}) Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, bc) \\
& + \sum_{\substack{i_1 < \dots < i_h = h+k \\ i_{h+1} < \dots < i_{h+k}}} \epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{h-1}}) + \bar{p}(bc))} \\
& \quad \times X(a_{i_1}, \dots, a_{i_{h-1}}, bc) Y(a_{i_h}, \dots, a_{i_{h+k}})
\end{aligned} \tag{3.6}$$

For the first summand in the right hand side, since $i_{h+k} = h+k$, we have:

$$\begin{aligned}
\epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) &= \epsilon_{a_1, \dots, a_{h+k-1}, b}(i_1, \dots, i_{h+k}) \\
&= \epsilon_{a_1, \dots, a_{h+k-1}, c}(i_1, \dots, i_{h+k}) \\
&= \epsilon_{a_1, \dots, a_{h+k-1}, 1}(i_1, \dots, i_{h+k})
\end{aligned}$$

and

$$\begin{aligned}
Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, bc) &= Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, b)c + (-1)^{p(b)p(c)} Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, c)b \\
&\quad - Y(a_{i_{h+1}}, \dots, a_{i_{h+k-1}}, 1)bc.
\end{aligned}$$

In the second summand, since $i_{h+k} = h$, we have:

$$\begin{aligned}
\epsilon_{a_1, \dots, a_{h+k-1}, bc}(i_1, \dots, i_{h+k}) &= \epsilon_{a_1, \dots, a_{h+k-1}, b}(i_1, \dots, i_{h+k}) (-1)^{p(c)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} \\
&= \epsilon_{a_1, \dots, a_{h+k-1}, c}(i_1, \dots, i_{h+k}) (-1)^{p(b)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} \\
&= \epsilon_{a_1, \dots, a_{h+k-1}, 1}(i_1, \dots, i_{h+k}) (-1)^{p(bc)(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))}
\end{aligned}$$

and

$$\begin{aligned}
& X(a_{i_1}, \dots, a_{i_{h-1}}, bc) Y(a_{i_h}, \dots, a_{i_{h+k}}) = \\
& (-1)^{p(c)(p(Y) + \bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} X(a_{i_1}, \dots, a_{i_{h-1}}, b) Y(a_{i_h}, \dots, a_{i_{h+k}}) c \\
& + (-1)^{p(b)(p(c) + p(Y) + \bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} X(a_{i_1}, \dots, a_{i_{h-1}}, c) Y(a_{i_h}, \dots, a_{i_{h+k}}) b \\
& - (-1)^{p(bc)(p(Y) + \bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{h+k}}))} X(a_{i_1}, \dots, a_{i_{h-1}}, 1) Y(a_{i_h}, \dots, a_{i_{h+k}}) bc
\end{aligned}$$

The generalized Leibniz rule for $X \wedge Y$ then follows by replacing these equalities in (3.6). \square

It remains to prove that the Lie bracket on $\Pi\mathcal{GW}^{as}(\Pi A)$ satisfies the generalized odd Leibniz rule (2.1). This follows from the following lemma.

Lemma 3.6 *The following equalities hold for $X, Y, Z \in \Pi\mathcal{GW}^{as}(\Pi A)$:*

$$\begin{aligned} X \square (Y \wedge Z) &= (X \square Y) \wedge Z + (-1)^{\bar{p}(X)p(Y)} Y \wedge (X \square Z) - (X \square 1) \wedge Y \wedge Z, \\ (X \wedge Y) \square Z &= X \wedge (Y \square Z) + (-1)^{p(Y)\bar{p}(Z)} (X \square Z) \wedge Y. \end{aligned}$$

Proof. An analogue result is proved in [5, Lemma 3.5]. For $X \in \Pi\mathcal{GW}_{l-k}^{as}(\Pi A)$, $Y \in \Pi\mathcal{GW}_{h-1}^{as}(\Pi A)$ and $Z \in \Pi\mathcal{GW}_{k-h-1}^{as}(\Pi A)$ with $h, k-h, l-k+1 \geq 0$, we have:

$$(3.7) \quad \begin{aligned} X \square (Y \wedge Z)(a_1, \dots, a_l) &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\ &\quad \times X(Y(a_{i_1}, \dots, a_{i_h})Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}) \end{aligned}$$

The generalized Leibniz rule for X can be rewritten in the following way:

$$\begin{aligned} X(bc, a_{k+1}, \dots, a_l) &= (-1)^{p(c)(\bar{p}(a_{k+1}) + \dots + \bar{p}(a_l))} X(b, a_{k+1}, \dots, a_l)c \\ &\quad + (-1)^{p(b)\bar{p}(X)} bX(c, a_{k+1}, \dots, a_l) \\ &\quad - (-1)^{p(bc)(\bar{p}(a_{k+1}) + \dots + \bar{p}(a_l))} X(1, a_{k+1}, \dots, a_l)bc. \end{aligned}$$

Using this equality in (3.7), $X \square (Y \wedge Z)(a_1, \dots, a_l)$ is then of the form:

$$X \square (Y \wedge Z)(a_1, \dots, a_l) = A + B - C.$$

The first term A is equal to

$$\begin{aligned} &\sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Z) + \bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\ &\quad \times X(Y(a_{i_1}, \dots, a_{i_h}), a_{i_{k+1}}, \dots, a_{i_l})Z(a_{i_{h+1}}, \dots, a_{i_k}) = \\ &= \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_{l-k+h} \\ i_{l-k+h+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{l-k+h}}))(\bar{p}(a_{i_{l-k+h+1}}) + \dots + \bar{p}(a_{i_l}))} \\ &\quad \times (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Z) + \bar{p}(a_{i_{l-k+h+1}}) + \dots + \bar{p}(a_{i_l}))(\bar{p}(a_{i_{h+1}}) + \dots + \bar{p}(a_{i_{l-k+h}}))} \\ &\quad \times X(Y(a_{i_1}, \dots, a_{i_h}), a_{i_{h+1}}, \dots, a_{i_{l-k+h}})Z(a_{i_{l-k+h+1}}, \dots, a_{i_l}) = \\ &= (X \square Y) \wedge Z(a_1, \dots, a_l). \end{aligned}$$

The second term B is equal to

$$\begin{aligned} &\sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Y) + \bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))\bar{p}(X)} \\ &\quad \times Y(a_{i_1}, \dots, a_{i_h})X(Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}) \\ &= (-1)^{p(Y)\bar{p}(X)} \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(p(Z) + \bar{p}(X))(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} \\ &\quad \times Y(a_{i_1}, \dots, a_{i_h})X(Z(a_{i_{h+1}}, \dots, a_{i_k}), a_{i_{k+1}}, \dots, a_{i_l}) = (-1)^{\bar{p}(X)p(Y)} Y \wedge (X \square Z)(a_1, \dots, a_l) \end{aligned}$$

since $p(X \square Z) = \bar{p}(X) + p(Z)$.

Finally, the third term C is equal to

$$\begin{aligned}
& \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}))} (-1)^{(p(Y) + p(Z) + \bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\
& \quad \times X(1, a_{i_{k+1}}, \dots, a_{i_l}) Y(a_{i_1}, \dots, a_{i_h}) Z(a_{i_{h+1}}, \dots, a_{i_k}) \\
& = \sum_{\substack{i_1 < \dots < i_h \\ i_{h+1} < \dots < i_k \\ i_{k+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_k}))(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\
& \quad \times (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_h}) + \bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} (-1)^{p(Y)(\bar{p}(a_{i_{k+1}}) + \dots + \bar{p}(a_{i_l}))} \\
& \quad \times X(1, a_{i_{k+1}}, \dots, a_{i_l}) Y(a_{i_1}, \dots, a_{i_h}) Z(a_{i_{h+1}}, \dots, a_{i_k}) = \\
& = \sum_{\substack{i_1 < \dots < i_{l-k} \\ i_{l-k+1} < \dots < i_{l-k+h} \\ i_{l-k+h+1} < \dots < i_l}} \epsilon_a(i_1, \dots, i_l) (-1)^{p(Z)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{l-k+h}}))} (-1)^{p(Y)(\bar{p}(a_{i_1}) + \dots + \bar{p}(a_{i_{l-k}}))} \\
& \quad \times (X \square 1)(a_{i_1}, \dots, a_{i_{l-k}}) Y(a_{i_{l-k+1}}, \dots, a_{i_{l-k+h}}) Z(a_{i_{l-k+h+1}}, \dots, a_{i_l}) = (X \square 1) \wedge Y \wedge Z(a_1, \dots, a_l).
\end{aligned}$$

This proves the first equality. The second equality can be proved in the same way, using the definition of the box product (3.2) and the concatenation product (3.5). \square

4 The main construction

Let $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \cdot)$ be a generalized n -Nambu-Poisson algebra and denote by $\Pi\mathcal{N}$ the space \mathcal{N} with reversed parity. Define

$$\begin{aligned}
(4.1) \quad & \mu : \Pi\mathcal{N} \otimes \dots \otimes \Pi\mathcal{N} \rightarrow \Pi\mathcal{N} \\
& \mu(f_1, \dots, f_n) = \{f_1, \dots, f_n\}.
\end{aligned}$$

Then μ is a supersymmetric function on $(\Pi\mathcal{N})^{\otimes n}$ [3, Lemma 1.2]. Furthermore μ satisfies the generalized Leibniz rule

$$\mu(f_1, \dots, f_{n-1}, gh) = \mu(f_1, \dots, f_{n-1}, g)h + g\mu(f_1, \dots, f_{n-1}, h) - \mu(f_1, \dots, f_{n-1}, 1)gh,$$

hence μ lies in $\mathcal{GW}_{n-1}^{as}(\Pi\mathcal{N})$.

Let $OP(\mathcal{N})$ be the odd Poisson subalgebra of $\mathcal{G}(\mathcal{N})$ generated by $\Pi\mathcal{N}$ and μ . Then, by construction, $OP(\mathcal{N})$ is a transitive Lie subalgebra of $\mathcal{GW}^{as}(\Pi\mathcal{N})$, hence it is a transitive subalgebra of $W(\Pi\mathcal{N})$. Furthermore $OP(\mathcal{N})$ is a \mathbb{Z}_+ -graded odd Poisson subalgebra of $\mathcal{G}(\mathcal{N})$. Let us denote by $OP(\mathcal{N}) = \oplus_{j \geq -1} \mathcal{P}_j(\mathcal{N})$ its depth 1 \mathbb{Z} -grading as a Lie superalgebra.

Proposition 4.1 *If \mathcal{N} is a simple generalized n -Nambu-Poisson algebra then $OP(\mathcal{N})$ is a simple generalized odd Poisson superalgebra.*

Proof. Let I be a non-zero ideal of $OP(\mathcal{N})$. Then, by transitivity, $I \cap \mathcal{P}_{-1}(\mathcal{N}) = I \cap \mathcal{N} \neq 0$. Note that $I \cap \mathcal{N}$ is a Nambu-Poisson ideal of \mathcal{N} . Indeed, $(I \cap \mathcal{N}) \cdot \mathcal{N} = (I \cap \mathcal{N}) \wedge \mathcal{N} \subset I \cap \mathcal{N}$ and $[I \cap \mathcal{N}, \mathcal{N}] \subset [\mathcal{N}, \mathcal{N}] = 0$. Since \mathcal{N} is simple, $I \cap \mathcal{N} = \mathcal{N}$, hence $1 \in I$, hence $I = OP(\mathcal{N})$. \square

Remark 4.2 We recall that since $(\mathcal{N}, \{\cdot, \dots, \cdot\})$ is an n -Lie algebra, the Filippov-Jacobi identity holds, i.e., for every $a_1, \dots, a_{n-1} \in \mathcal{N}$, the map $D_{a_1, \dots, a_{n-1}} : \mathcal{N} \rightarrow \mathcal{N}$, $D_{a_1, \dots, a_{n-1}}(a) = \{a_1, \dots, a_{n-1}, a\}$ is a derivation of $(\mathcal{N}, \{\cdot, \dots, \cdot\})$. By [4, Lemma 2.1(b)], this is equivalent to the condition $[\mu, D_{a_1, \dots, a_{n-1}}] = 0$ in $OP(\mathcal{N})$. By (4.1), we have: $D_{a_1, \dots, a_{n-1}} = [[\mu, a_1], \dots, a_{n-1}]$, therefore μ satisfies the following condition:

$$[\mu, [[\mu, a_1], \dots, a_{n-1}]] = 0 \quad \text{for every } a_1, \dots, a_{n-1} \in \mathcal{N}.$$

Definition 4.3 We say that a pair (\mathcal{P}, μ) , consisting of a \mathbb{Z}_+ -graded generalized odd Poisson superalgebra \mathcal{P} and an element $\mu \in \mathcal{P}_{n-1}$ of parity $p(\mu) \equiv n \pmod{2}$, is a good n -pair if it satisfies the following properties:

- G1) $\mathcal{P} = \bigoplus_{j \geq -1} \mathcal{P}_j$ is a transitive \mathbb{Z} -graded Lie superalgebra of depth 1 such that \mathcal{P}_{-1} is completely odd;
- G2) μ and \mathcal{P}_{-1} generate \mathcal{P} as a (generalized) odd Poisson superalgebra;
- G3) $[\mu, [[\mu, a_1], \dots, a_{n-1}]] = 0$ for every $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1}$.

Example 4.4 Let $\mathcal{P} = PO(2h, 2h)$, $h \geq 1$, with the grading of type $(0, \dots, 0|1, \dots, 1)$, and let $\mu = \sum_{i=1}^h \xi_i \xi_{i+h}$. Then (\mathcal{P}, μ) is a good 2-pair. Indeed, for $1 \leq i \leq h$, $[x_i, \mu]_{HO} = \xi_{h+i}$ and $[x_{h+i}, \mu]_{HO} = -\xi_i$, therefore \mathcal{P}_{-1} and μ generate \mathcal{P} . Furthermore, for $f \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$, we have: $[f, \mu]_{HO} = \sum_{i=1}^h (\frac{\partial f}{\partial x_i} \xi_{i+h} - \frac{\partial f}{\partial x_{i+h}} \xi_i)$, hence

$$\begin{aligned} [\mu, [f, \mu]_{HO}]_{HO} &= \sum_{i,j=1}^h [\xi_j \xi_{j+h}, \frac{\partial f}{\partial x_i} \xi_{i+h} - \frac{\partial f}{\partial x_{i+h}} \xi_i]_{HO} = \\ &= \sum_{i,j=1}^h (\frac{\partial^2 f}{\partial x_i \partial x_j} \xi_{j+h} \xi_{i+h} - \frac{\partial^2 f}{\partial x_{j+h} \partial x_i} \xi_j \xi_{i+h} - \frac{\partial^2 f}{\partial x_j \partial x_{i+h}} \xi_{j+h} \xi_i + \frac{\partial^2 f}{\partial x_{i+h} \partial x_{j+h}} \xi_j \xi_i) = 0. \end{aligned}$$

Therefore (\mathcal{P}, μ) satisfies property G3).

Example 4.5 Let $\mathcal{P} = PO(n, n)$ with the grading of type $(0, \dots, 0|1, \dots, 1)$, and let $\mu = \xi_1 \dots \xi_n$. Then (\mathcal{P}, μ) is a good n -pair. Indeed, $[x_{n-1}, [\dots, [x_2, [x_1, \mu]]]]_{HO} = \xi_n$, and, similarly all the ξ_i 's can be obtained by commuting μ with different x_j 's. Therefore \mathcal{P}_{-1} and μ generate \mathcal{P} . Furthermore, let $f = \sum_{i=1}^n f_i \xi_i \in \mathcal{P}_0$, with $f_i \in \mathbb{F}[[x_1, \dots, x_n]]$, such that

$$(4.2) \quad \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} = 0.$$

Then $[f, \mu]_{HO} = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \xi_1 \dots \xi_n = 0$. Notice that all elements of the form $[[\mu, a_1], \dots, a_{n-1}]$ with $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$ satisfy property (4.2), hence (\mathcal{P}, μ) satisfies property G3).

Example 4.6 Let $\mathcal{P} = PO(2h+1, 2h+2)$, $h \geq 1$, with the grading of type $(0, \dots, 0|1, \dots, 1, 1)$, and let $\mu = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$ (recall that $\xi_{2h+2} = \tau$). Then (\mathcal{P}, μ) is a good 2-pair. Indeed, we have: $[1, \mu]_{KO} = 2\xi_{h+1}$ and $[x_i, \mu]_{KO} = \xi_{i+h+1} - x_i \xi_{h+1}$ for $1 \leq i \leq h+1$, $[x_{i+h+1}, \mu]_{KO} = -\xi_i - x_{i+h+1} \xi_{h+1}$ for $1 \leq i \leq h$. Hence \mathcal{P}_{-1} and μ generate \mathcal{P} . Furthermore, if $f \in \mathcal{P}_{-1} = \mathbb{F}[[x_1, \dots, x_n]]$, we have:

$$[f, \mu]_{KO} = \sum_{i=1}^h \left(\frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} - (E-2)(f) \xi_{h+1},$$

hence

$$\begin{aligned} [\mu, [f, \mu]_{KO}]_{KO} &= \left[\sum_{j=1}^{h+1} \xi_j \xi_{h+1+j}, \sum_{i=1}^h \left(\frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} - (E-2)(f) \xi_{h+1} \right]_{KO} \\ &= \sum_{i=1, \dots, h; j=1, \dots, h+1} \xi_{h+1+j} \left(\frac{\partial^2 f}{\partial x_j \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_j \partial x_{i+h+1}} \xi_i \right) + \sum_{j=1}^{h+1} \xi_{h+1+j} \left(\frac{\partial^2 f}{\partial x_j \partial x_{h+1}} \xi_{2h+2} \right. \\ &\quad \left. - \frac{\partial((E-2)(f))}{\partial x_j} \xi_{h+1} \right) - \sum_{i,j=1}^h \xi_j \left(\frac{\partial^2 f}{\partial x_{h+1+j} \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_{h+1+j} \partial x_{i+h+1}} \xi_i \right) - \sum_{j=1}^h \xi_j \left(\frac{\partial^2 f}{\partial x_{h+1+j} \partial x_{h+1}} \xi_{2h+2} \right. \\ &\quad \left. - \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_{h+1} \right) - \xi_{h+1} (E-2)([f, \mu]_{KO}) = \xi_{2h+2} \sum_{i=1}^h \left(\frac{\partial^2 f}{\partial x_{h+1} \partial x_i} \xi_{h+i+1} - \frac{\partial^2 f}{\partial x_{h+1} \partial x_{i+h+1}} \xi_i \right) \\ &\quad + \sum_{j=1}^{h+1} \frac{\partial^2 f}{\partial x_j \partial x_{h+1}} \xi_{h+1+j} \xi_{2h+2} - \sum_{j=1}^{h+1} \xi_{h+1+j} \frac{\partial((E-2)(f))}{\partial x_j} \xi_{h+1} - \sum_{j=1}^h \left(\frac{\partial^2 f}{\partial x_{h+j+1} \partial x_{h+1}} \xi_j \xi_{2h+2} \right. \\ &\quad \left. - \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_j \xi_{h+1} \right) - \xi_{h+1} (E-2)([f, \mu]_{KO}) = \xi_{h+1} \left(\sum_{j=1}^{h+1} \xi_{h+1+j} \frac{\partial((E-2)(f))}{\partial x_j} \right. \\ &\quad \left. - \sum_{j=1}^h \frac{\partial((E-2)(f))}{\partial x_{h+1+j}} \xi_j - (E-2) \left(\sum_{i=1}^h \left(\frac{\partial f}{\partial x_i} \xi_{i+h+1} - \frac{\partial f}{\partial x_{i+h+1}} \xi_i \right) + \frac{\partial f}{\partial x_{h+1}} \xi_{2h+2} \right) \right) = 0. \end{aligned}$$

Example 4.7 Let $\mathcal{P} = PO(n, n+1)$, with the grading of type $(0, \dots, 0|1, \dots, 1, 1)$, and let $\mu = \xi_1 \dots \xi_n \tau$ (recall that $\tau = \xi_{n+1}$). Then (\mathcal{P}, μ) is a good $(n+1)$ -pair. Indeed, we have: $[1, \mu]_{KO} = 2(-1)^{n+1} \xi_1 \dots \xi_n$, $[x_{i_1}, [\dots, [x_{i_{n-1}}, \xi_1 \dots \xi_n]_{KO}]_{KO}]_{KO} = \pm \xi_{i_n}$ for $i_1 \neq \dots \neq i_{n-1} \neq i_n$, $[x_i, \xi_i \dots \xi_n \tau]_{KO} = \xi_{i+1} \dots \xi_n \tau + (-1)^{n-i} x_i \xi_i \dots \xi_n$ for $1 \leq i \leq n$. Hence \mathcal{P}_{-1} and μ generate \mathcal{P} . Now let $\text{div}_1 = \Delta + (E-n) \frac{\partial}{\partial \tau}$ where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial \xi_i}$ is the odd Laplacian, and let $f = \sum_{i=1}^{n+1} f_i \xi_i \in \mathcal{P}_0$, $f_i \in \mathbb{F}[[x_1, \dots, x_n]]$, such that $0 = \text{div}_1(f) = \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} + (E-n)(f_{n+1})$. Then we have:

$$\begin{aligned} \left[\sum_{i=1}^{n+1} f_i \xi_i, \mu \right]_{KO} &= \left[\sum_{i=1}^{n+1} f_i \xi_i, \mu \right]_{HO} + \sum_{i=1}^{n+1} (E-2)(f_i \xi_i) (-1)^n \xi_1 \dots \xi_n - f_{n+1} (n-2) \xi_1 \dots \xi_n \tau \\ &= \sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} \xi_1 \dots \xi_n \tau + (-1)^n (E-2)(f_{n+1} \xi_{n+1}) \xi_1 \dots \xi_n - (n-2) f_{n+1} \xi_1 \dots \xi_n \tau \end{aligned}$$

$$= \left(\sum_{i,j=1}^n \frac{\partial f_i}{\partial x_j} + (E-2)(f_{n+1}) - (n-2)f_{n+1}\xi_1 \dots \xi_n \tau \right) = 0.$$

Notice that, since $\text{div}_1(\mu) = 0$ and $\text{div}_1(f) = 0$ for every $f \in \mathcal{P}_{-1}$, then $\text{div}_1([\mu, a_1], \dots, a_{n-1}]) = 0$ for every $a_1, \dots, a_{n-1} \in \mathcal{P}_{-1}$. Hence property *G3*) is satisfied.

Remark 4.8 Let us consider $\mathcal{P} = PO(k, k)$ (resp. $\mathcal{P} = PO(k, k+1)$) with the grading of type $(0, \dots, 0|1, \dots, 1)$ (resp. $(0, \dots, 0|1, \dots, 1, 1)$). Let $\varphi \in \mathcal{P}_{-1}$ be an invertible element. By Remark 2.9, the grading of type $(0, \dots, 0|1, \dots, 1)$ (resp. $(0, \dots, 0|1, \dots, 1, 1)$) defines a \mathbb{Z}_+ -graded structure on the odd generalized Poisson superalgebra \mathcal{P}^φ , such that $\mathcal{P}_j = \mathcal{P}_j^\varphi$. Then, by (2.4), (\mathcal{P}, μ) is a good n -pair with respect to this grading if and only if $(\mathcal{P}^\varphi, \varphi^{-1}\mu)$ is.

The map $\mathcal{N} \mapsto (OP(\mathcal{N}), \mu)$ establishes a correspondence between (simple) generalized n -Nambu-Poisson algebras \mathcal{N} and good n -pairs $(OP(\mathcal{N}), \mu)$. We now want to show that this correspondence is bijective.

Lemma 4.9 *Let \mathcal{N} be a generalized n -Nambu-Poisson algebra. Then the 0-th graded component $\mathcal{P}_0(N)$ of $OP(N)$ is generated, as a Lie superalgebra, by elements of the form*

$$[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b$$

with $a_i, b \in \Pi\mathcal{N}$.

Proof. Let $L_{-1} := \Pi\mathcal{N}$ and let L_0 be the Lie subsuperalgebra of $\mathcal{GW}_0^{as}(\Pi\mathcal{N}) = GDer(\Pi\mathcal{N})$ generated by the elements of the form $[a_1, [a_2, \dots, [a_{n-1}, \mu b]]]$ with $a_1, \dots, a_{n-1}, b \in \Pi\mathcal{N}$. Note that, since $\mathcal{GW}_0^{as}(\Pi\mathcal{N})$ is \mathbb{Z} -graded of depth 1, and $1 \in \mathcal{N}$, the restriction to \mathcal{N} of the derivation D of $\mathcal{G}(\mathcal{N})$ is zero, hence

$$[a_1, [a_2, \dots, [a_{n-1}, \mu b]]] = [a_1, [a_2, \dots, [a_{n-1}, \mu]]]b.$$

An induction argument on the length of the commutators of the generating elements of L_0 shows that L_0 is stable with respect to the concatenation product by elements of $\Pi\mathcal{N}$.

Let L be the full prolongation of $L_{-1} \oplus L_0$, i.e., $L = L_{-1} \oplus L_0 \oplus (\oplus_{j \geq 1} L_j)$, where $L_j = \{\varphi \in \mathcal{GW}^{as}(\Pi\mathcal{N}) \mid [\varphi, L_{-1}] \subset L_{j-1}\}$. Note that L_j , for $j \geq 1$, is stable with respect to the concatenation product by elements of $\Pi\mathcal{N}$. Indeed, if $\varphi \in L_j$, then

$$[\varphi \Pi\mathcal{N}, L_{-1}] = [\varphi, L_{-1}] \Pi\mathcal{N} \subset L_{j-1} \Pi\mathcal{N},$$

hence one can conclude by induction on j since $L_0 \Pi\mathcal{N} \subset L_0$. It follows that L is closed under the concatenation product, hence it is an odd generalized Poisson subsuperalgebra of $\mathcal{GW}^{as}(\Pi\mathcal{N})$. Indeed, using induction on $i+j \geq 0$, one shows that $L_i L_j \subset L$ for every $i, j \geq 0$.

It follows that $OP(\mathcal{N})$ is an odd generalized subsuperalgebra of L , since L is an odd generalized Poisson superalgebra containing $\Pi\mathcal{N}$ and μ . As a consequence, the 0-th graded component $\mathcal{P}_0(\mathcal{N})$ of $OP(\mathcal{N})$ is generated, as a Lie superalgebra, by elements of the form

$$[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b$$

with $a_i, b \in \Pi\mathcal{N}$. □

Proposition 4.10 *Let (\mathcal{P}, μ) be a good n -pair, and define on $\mathcal{N} := \Pi\mathcal{P}_{-1}$ the following product:*

$$\{x_1, \dots, x_n\} = [\dots [[\mu, x_1], \dots, x_n]].$$

Then:

- (a) $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \wedge)$ is a generalized Nambu-Poisson algebra, \wedge being the restriction to \mathcal{N} of the commutative associative product \wedge defined on \mathcal{P} .
- (b) If \mathcal{P} is a simple odd generalized Poisson superalgebra, then $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \wedge)$ is a simple generalized Nambu-Poisson algebra.

Proof. (a) By Definitions 2.1 and 2.6, $\mathcal{N} = \mathcal{Q}_0$ is a commutative associative subalgebra of \mathcal{P} . Furthermore $\{\cdot, \dots, \cdot\}$ is an n -Lie bracket due to [4, Prop. 2.4] and property G3). Finally, for $f_1, \dots, f_{n-1}, g, h \in \Pi\mathcal{P}_{-1}$, we have:

$$\begin{aligned} \{f_1, \dots, f_{n-1}, gh\} &= [[\dots [\mu, f_1], \dots, f_{n-1}], gh] = [[\dots [\mu, f_1], \dots, f_{n-1}], g]h + g[[\dots [\mu, f_1], \dots, f_{n-1}], h] \\ &+ (-1)^{p([\dots [\mu, f_1], \dots, f_{n-1}]) + 1} [1, [\dots [\mu, f_1], \dots, f_{n-1}]]gh = \{f_1, \dots, f_{n-1}, g\}h + g\{f_1, \dots, f_{n-1}, h\} \\ &- (-1)^{p([\dots [\mu, f_1], \dots, f_{n-1}]) + 1} (-1)^{\bar{p}([\dots [\mu, f_1], \dots, f_{n-1}])} \{f_1, \dots, f_{n-1}, 1\}gh \\ &= \{f_1, \dots, f_{n-1}, g\}h + g\{f_1, \dots, f_{n-1}, h\} - \{f_1, \dots, f_{n-1}, 1\}gh. \end{aligned}$$

(b) Now we want to show that if \mathcal{P} is simple, then \mathcal{N} is simple. Suppose that I is a non zero ideal of \mathcal{N} , and let \tilde{I} be the ideal of \mathcal{P} generated by ΠI and μ : $\tilde{I} = \oplus_{j \geq -1} \tilde{I}_j$, with $\tilde{I}_j \subset \mathcal{P}_j$. We want to show that $\tilde{I}_{-1} = \tilde{I} \cap \mathcal{P}_{-1} = \Pi I$. In fact, the concatenation product by elements in $\oplus_{j \geq 1} \mathcal{Q}_j$ maps \mathcal{Q}_0 to $\oplus_{j \geq 1} \mathcal{Q}_j$ hence it does not produce any element in $\mathcal{P}_{-1} = \mathcal{Q}_0$. On the other hand, $I \wedge \mathcal{Q}_0 = I \wedge \mathcal{N} \subset I$ since I is an ideal of \mathcal{N} . The bracket between elements in $\oplus_{j \geq 0} \mathcal{P}_j$ lies in $\oplus_{j \geq 0} \mathcal{P}_j$ and the bracket between I and elements in $\oplus_{j \geq 1} \mathcal{P}_j$ lies in $\oplus_{j \geq 0} \mathcal{P}_j$. Therefore we just need to consider the brackets between elements in I and elements in \mathcal{P}_0 . By hypothesis, \mathcal{P} is generated by \mathcal{P}_{-1} and μ , hence, by the same argument as in Lemma 4.9, \mathcal{P}_0 is generated by elements of the form $[a_1, [a_2, \dots, [a_{n-1}, \mu]]]b$ with $a_i, b \in \Pi\mathcal{P}_{-1}$. We have:

$$[I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]b] = [I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]]b$$

since $[I, b] = 0$ and $D|_I = 0$. Since $[I, [a_1, [a_2, \dots, [a_{n-1}, \mu]]]] = \{I, a_1, \dots, a_{n-1}\}$ and I is an ideal of \mathcal{N} , $[I, \mathcal{P}_0] \subset I$. \square

Definition 4.11 *Two good n -pairs (\mathcal{P}, μ) and (\mathcal{P}', μ') are called isomorphic if there exists an odd Poisson superalgebras isomorphism $\Phi : \mathcal{P} \rightarrow \mathcal{P}'$ such that $\Phi(\mathcal{P}_j) = \mathcal{P}'_j$, $\Phi(\mathcal{Q}_j) = \mathcal{Q}'_j$ for all j and $\phi(\mu) \in \mathbb{F}^\times \mu'$.*

Theorem 4.12 *The map*

$$\mathcal{N} \rightarrow (OP(\mathcal{N}), \mu)$$

with μ defined as in (4.1), establishes a bijection between isomorphism classes of generalized n -Nambu-Poisson algebras and isomorphism classes of good n -pairs. Moreover:

- (i) \mathcal{N} is simple (linearly compact) if and only if $OP(\mathcal{N})$ is;
- (ii) \mathcal{N} is a Nambu-Poisson algebra if and only if $OP(\mathcal{N})$ is an odd Poisson superalgebra.

Proof. The proof follows immediately from Propositions 4.1 and 4.10. The fact that the linear compactness of \mathcal{N} implies that of $OP(\mathcal{N})$ can be proved in the same way as in [4, Proposition 2.4]. \square

Remark 4.13 One can check (see also [4]) that if \mathcal{N} is the n -Nambu algebra, then $(OP(\mathcal{N}), \mu) = (PO(n, n), \xi_1 \dots \xi_n)$ and if \mathcal{N} is the n -Dzhumaldidaev algebra, then $(OP(\mathcal{N}), \mu) = (PO(n - 1, n), \xi_1 \dots \xi_{n-1} \tau)$.

5 Classification of good pairs

In this section we will consider the odd Poisson (resp. generalized odd Poisson) superalgebra $PO(n, n)$ (resp. $PO(n, n+1)$) with the grading of type $(0, \dots, 0|1, \dots, 1)$ (resp. $(0, \dots, 0|1, \dots, 1, 1)$).

Proposition 5.1 *Let $\mathcal{P} = PO(n, n)$ or $\mathcal{P} = PO(n, n+1)$ and (\mathcal{P}, μ) be a good k -pair. Then the Lie subalgebra \mathcal{P}_0 of \mathcal{P} is spanned by elements of the form:*

$$[[\mu, a_1], \dots, a_{k-1}]b$$

with $a_1, \dots, a_{k-1}, b \in \mathcal{P}_{-1}$.

Proof. By Theorem 4.12, $\mathcal{P} = OP(\mathcal{N})$ for some k -Nambu-Poisson algebra \mathcal{N} . Hence, by Lemma 4.9, \mathcal{P}_0 is generated as a Lie algebra by elements of the form

$$[[\mu, a_1], \dots, a_{k-1}]b$$

with $a_1, \dots, a_{k-1}, b \in \mathcal{P}_{-1}$. Let $S = \langle [[\mu, a_1], \dots, a_{k-1}] \mid a_1, \dots, a_{k-1} \in \mathcal{P}_{-1} \rangle \subset \mathcal{P}_0$.

Let $\mathcal{P} = PO(n, n)$. Then, for $z_1, z_2 \in S$, $b_1, b_2 \in \mathcal{P}_{-1}$, we have:

$$\begin{aligned} [z_1 b_1, z_2 b_2] &= [z_1 b_1, z_2] b_2 + (-1)^{p(z_2)(p(z_1)+p(b_1)+1)} z_2 [z_1 b_1, b_2] = (-1)^{p(b_1)(p(z_2)+1)} [z_1, z_2] b_1 b_2 + \\ &\quad + z_1 [b_1, z_2] b_2 + (-1)^{p(b_1)(p(b_2)+1)+p(z_2)(p(z_1)+p(b_1)+1)} z_2 [z_1, b_2] b_1 \end{aligned}$$

since $[b_1, b_2] = 0$. We recall that $[z_1, z_2]$ lies in S by [4, Theorem 0.2]. Finally, note that $[z_1, b_2]$ and $[b_1, z_2]$ lie in \mathcal{P}_{-1} . It follows that $\mathcal{P}_0 \subseteq \langle [[\mu, a_1], \dots, a_{k-1}]b \mid a_i, b \in \mathcal{P}_{-1} \rangle \subseteq \mathcal{P}_0$, hence the statement holds for $\mathcal{P} = PO(n, n)$.

If $\mathcal{P} = PO(n, n+1)$, one uses exactly the same argument and the fact that $D|_{\mathcal{P}_{-1}} = 0$, $D(S) \subseteq \mathcal{P}_{-1}$. \square

For any element $f \in \mathcal{P}_{k-1} = \mathbb{F}[[x_1, \dots, x_n]] \otimes \wedge^k \mathbb{F}^n$, we let $f_0 = f|_{x_1=\dots=x_n=0} \in \wedge^k \mathbb{F}^n$. We shall say that f has positive order if $f_0 = 0$.

Corollary 5.2 *Let $\mathcal{P} = PO(n, n)$ (resp. $PO(n, n+1)$) with the grading of type $(0, \dots, 0|1, \dots, 1)$ (resp. $(0, \dots, 0|1, \dots, 1, 1)$). If $\mu \in \mathcal{P}_{k-1}$ is such that μ_0 lies in the Grassmann subalgebra of $\wedge^k(\mathbb{F}^n)$ (resp. $\wedge^k(\mathbb{F}^{n+1})$) generated by some variables $\xi_{i_1}, \dots, \xi_{i_h}$, for some $h < n$ (resp. $h < n+1$), then μ does not satisfy property G2). In particular, if $\mu_0 = 0$, then μ does not satisfy property G2).*

Proof. Suppose, on the contrary, that some ξ_i does not appear in the expression of μ_0 . Then, by Proposition 5.1, \mathcal{P}_0 does not contain ξ_i and this is a contradiction since if $\mathcal{P} = PO(n, n)$ (resp. $\mathcal{P} = PO(n, n+1)$), $\mathcal{P}_0 = \langle \xi_1, \dots, \xi_n \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]]$ (resp. $\mathcal{P}_0 = \langle \xi_1, \dots, \xi_{n+1} \rangle \otimes \mathbb{F}[[x_1, \dots, x_n]]$). \square

5.1 The case $PO(n, n)$

In this subsection we shall determine good k -pairs (\mathcal{P}, μ) for $\mathcal{P} = PO(n, n)$ with the \mathbb{Z}_+ -grading of type $(0, \dots, 0 | 1, \dots, 1)$. We will denote the Lie superalgebra bracket in $PO(n, n)$ simply by $[\cdot, \cdot]$. Recall the corresponding description of the \mathbb{Z}_+ -grading given in Example 2.7. When writing a monomial in ξ_i 's we will assume that the indices increase; elements from $\wedge^k \mathbb{F}^n$ will be written as linear combinations of such monomials.

Lemma 5.3 *Let $2 < k < n - 1$ and suppose that $\mu \in PO(n, n)_{k-1}$ can be written in the following form:*

$$(5.1) \quad \mu = \xi_1 \dots \xi_k + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi + \psi,$$

where:

$$\mu_0 = \xi_1 \dots \xi_k + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi, \quad \varphi \in \wedge^k \mathbb{F}^n, \quad \psi_0 = 0,$$

$$h = \max\{0 \leq j \leq k - 2 \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and some } r, s > k\},$$

$$\frac{\partial^{k-1} \varphi}{\partial \xi_1 \dots \partial \xi_{k-1}} = 0, \quad \frac{\partial^k \varphi}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \xi_{k+2} \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-2}}} = 0.$$

Then μ does not satisfy property G3).

Proof. Let us first suppose that $h \geq 1$. We have:

$$[x_{k+1}, \mu] = (-1)^h \xi_1 \dots \xi_h \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \frac{\partial(\varphi + \psi)}{\partial \xi_{k+1}};$$

$$[x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] = 2(-1)^{k-2} x_1 \xi_{k+2} + 2x_1 \frac{\partial^{k-1}(\varphi + \psi)}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}}.$$

Therefore $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] =$

$$= 2(-1)^k (\xi_2 \dots \xi_k ((-1)^{k-2} \xi_{k+2} + \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}}) +$$

$$+ \xi_2 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}} +$$

$$\frac{\partial \varphi}{\partial \xi_1} ((-1)^{k-2} \xi_{k+2} + \frac{\partial^{k-1} \varphi}{\partial \xi_{i_{k-2}} \dots \partial \xi_{i_{h+1}} \partial \xi_h \dots \partial \xi_1 \partial \xi_{k+1}})) + \omega,$$

for some ω of positive order. Note that, the summand $2\xi_2 \dots \xi_k \xi_{k+2}$ in the expression of $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]]$ does not cancel out. Indeed, due to the hypotheses on φ , the only possibility to cancel the summand $2\xi_2 \dots \xi_k \xi_{k+2}$ is that the expression of φ contains the sum $a\xi_1 \dots \xi_h \xi_{k+1} \xi_{i_{h+1}} \dots \xi_{i_{k-2}} \xi_t + b\xi_1 \dots \xi_{t-1} \xi_{t+1} \dots \xi_k \xi_{k+2}$, for some t , $2 \leq t \leq k$, and some suitable coefficients $a, b \in \mathbb{F}^*$. But this is impossible since it is in contradiction with the maximality of h if $h = k - 2$, and with the hypotheses on φ if $h < k - 2$. It follows that $[\mu, [x_{i_{k-2}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] \neq 0$ and property G3) is not satisfied.

If $h = 0$, then one can use the same argument by showing that the commutator

$$[\mu, [x_1 x_{k+1}, [x_{i_1}, \dots, [x_{i_{k-2}}, \mu]]]]$$

□

Theorem 5.4 *Let $\mathcal{P} = PO(n, n)$. Suppose that $2 < k < n - 1$ and that $\mu \in PO(n, n)_{k-1}$. Then (\mathcal{P}, μ) is not a good k -pair.*

Proof. By Corollary 5.2, if $\mu_0 = 0$ then μ does not satisfy property G2). Now suppose $\mu_0 \neq 0$. Since μ_0 lies in $\wedge^k(\mathbb{F}^n)$, we can assume, up to a linear change of indeterminates, that $\mu_0 = \xi_1 \dots \xi_k + f$ for some $f \in \wedge^k(\mathbb{F}^n)$ such that $\frac{\partial^k f}{\partial \xi_1 \dots \partial \xi_k} = 0$. Then, either μ does not satisfy property G2) and (\mathcal{P}, μ) is not a good k -pair, or, again by Corollary 5.2, all ξ_i 's appear in the expression of μ_0 . Let us thus assume to be in the latter case. Then, since $k < n - 1$, either there exist some $r, s > k$ such that the indeterminates ξ_r and ξ_s both appear in the expression of μ_0 in at least one monomial (case A), or all the indeterminates ξ_r and ξ_s with $r, s > k$ appear in distinct monomials (case B).

Suppose we are in case A), and let $h = \max\{0 \leq j \leq k - 2 \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, i_1 < \dots < i_j \leq k; r, s > k\}$. Then we can write

$$\mu_0 = \xi_1 \dots \xi_k + \xi_{i_1} \dots \xi_{i_h} \xi_r \xi_s \xi_{i_{h+1}} \dots \xi_{i_{k-2}} + \varphi$$

for some $r, s, i_{h+1}, \dots, i_{k-2} > k$, $i_1, \dots, i_h \leq k$ and some $\varphi \in \wedge^k(\mathbb{F}^n)$ such that $\frac{\partial^k \varphi}{\partial \xi_1 \dots \partial \xi_k} = 0$ and $\frac{\partial^k \varphi}{\partial \xi_{i_1} \dots \partial \xi_{i_h} \partial \xi_r \partial \xi_s \partial \xi_{i_1} \dots \partial \xi_{i_{k-2}}} = 0$. Up to a permutation of indices we can assume $r = k + 1$, $s = k + 2$, $\{i_1, \dots, i_h\} = \{1, \dots, h\}$ and up to a linear change of indeterminates we can assume $\frac{\partial^{k-1} \varphi}{\partial \xi_1 \dots \partial \xi_{k-1}} = 0$. Therefore μ satisfies the hypotheses of Lemma 5.3, hence it does not satisfy property G3).

Now suppose we are in case B). Then

$$\mu_0 = \xi_1 \dots \xi_k + \xi_{i_1} \dots \xi_{i_{k-1}} \xi_{k+1} + \xi_{j_1} \dots \xi_{j_{k-1}} \xi_{k+2} + \psi$$

for some $i_1 < \dots < i_{k-1} \leq k$, $j_1 < \dots < j_{k-1} \leq k$ and $\psi \in \wedge^k(\mathbb{F}^n)$ such that $\frac{\partial^k \psi}{\partial \xi_1 \dots \partial \xi_k} = 0$, $\frac{\partial^k \psi}{\partial \xi_{i_1} \dots \partial \xi_{i_{k-1}} \partial \xi_{k+1}} = 0$, $\frac{\partial^k \psi}{\partial \xi_{j_1} \dots \partial \xi_{j_{k-1}} \partial \xi_{k+2}} = 0$, $\frac{\partial^2 \psi}{\partial \xi_r \partial \xi_s} = 0$ for every $r, s > k$. Again by Corollary 5.2, we can assume that $\{i_1, \dots, i_{k-1}\} \neq \{j_1, \dots, j_{k-1}\} \neq \{1, \dots, k - 1\}$. Therefore there exists an index $j_l \in \{1, \dots, k\} \cap \{j_1, \dots, j_{k-1}\}$ such that $j_l \notin \{i_1, \dots, i_{k-1}\}$.

Now consider the following change of indeterminates:

$$\xi'_{j_l} = \xi_{j_l} + \xi_{k+1}; \quad \xi'_j = \xi_j \quad \forall j \neq j_l.$$

Then

$$\mu_0 = \xi'_1 \dots \xi'_k + \xi'_{j_1} \dots \xi'_{j_{k-1}} \xi'_{k+2} + \xi'_{j_1} \dots \xi'_{j_{i-1}} \xi'_{j_{i+1}} \dots \xi'_{j_{k-1}} \xi'_{k+1} \xi'_{k+2} + \rho$$

for some $\rho \in \wedge^k(\mathbb{F}^n)$ such that $\frac{\partial^k \rho}{\partial \xi'_1 \dots \partial \xi'_k} = 0$, $\frac{\partial^k \rho}{\partial \xi'_{j_1} \dots \partial \xi'_{j_{k-1}} \partial \xi'_{k+2}} = 0$, $\frac{\partial^k \rho}{\partial \xi'_{j_1} \dots \partial \xi'_{j_{i-1}} \partial \xi'_{j_{i+1}} \dots \partial \xi'_{j_{k-1}} \partial \xi'_{k+1} \partial \xi'_{k+2}} = 0$.

We are now again in case A) hence the proof is concluded. \square

Theorem 5.5 *Let $\mathcal{P} = PO(n, n)$. If (\mathcal{P}, μ) is a good k -pair, then, up to isomorphisms, one of the following possibilities may occur:*

a) If $n = 2h$:

$$a1) \quad k = 2 \text{ and } \mu_0 = \sum_{i=1}^h \xi_i \xi_{i+h};$$

$$a2) \quad k = n \text{ and } \mu_0 = \xi_1 \dots \xi_n.$$

b) If $n = 2h + 1$:

$$b1) \quad k = n \text{ and } \mu_0 = \xi_1 \dots \xi_n.$$

Proof. By Theorem 5.4, the only possibilities for k are $k = 2$, $k = n - 1$ or $k = n$.

By Corollary 5.2, $\frac{\partial \mu_0}{\partial \xi_i} \neq 0$ for every $i = 1, \dots, n$. Using the classification of non-degenerate skew-symmetric bilinear forms, it thus follows that the case $k = 2$ can occur only if $n = 2h$ and, up to equivalence, $\mu_0 = \sum_{i=1}^h \xi_i \xi_{i+h}$, hence we get a1).

If $k = n$ then, up to rescaling the odd indeterminates, $\mu_0 = \xi_1 \dots \xi_n$ and we get cases a2) and b1).

Now assume $k = n - 1$. Assume that $\frac{\partial^{n-2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_{n-2}}} = \alpha \xi_{i_{n-1}} + \beta \xi_{i_n}$ for some $i_1 < \dots < i_{n-2}$, $i_{n-1} < i_n$, and some $\alpha, \beta \in \mathbb{F}^*$. Consider the following change of indeterminates:

$$\xi'_{i_{n-1}} = \alpha \xi_{i_{n-1}} + \beta \xi_{i_n} \quad \xi'_{i_j} = \xi_{i_j} \quad \forall j \neq n-1.$$

Then $\frac{\partial^{n-2} \mu_0}{\partial \xi'_{i_1} \dots \partial \xi'_{i_{n-2}}} = \xi'_{i_{n-1}}$. By using induction on the lexicographic order of the indices $i_1 < \dots < i_{n-2}$, one can thus show that, up to a linear change of indeterminates, $\mu_0 = \xi_1 \dots \xi_{n-1}$, hence (\mathcal{P}, μ) is not a good k -pair due to Corollary 5.2. \square

5.2 The case $PO(n, n+1)$

In this subsection we shall determine good pairs (\mathcal{P}, μ) for $\mathcal{P} = PO(n, n+1)$ with the \mathbb{Z} -grading of type $(0, \dots, 0|1, \dots, 1, 1)$. We shall adopt the same notation as in the previous subsection.

Lemma 5.6 *Let $2 \leq k < n - 1$, $\mu \in PO(n, n+1)_k$ and suppose that μ_0 can be written in one of the following forms:*

1.

$$(5.2) \quad \mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi$$

where:

$$(a) \quad h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and } r, s > k\};$$

$$(b) \quad \varphi \in \wedge^{k+1} \mathbb{F}^{n+1} \text{ is such that } \frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0;$$

2.

$$(5.3) \quad \mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \tau \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi$$

where:

$$(a) \quad h = \max\{0 \leq j < k \mid \frac{\partial^{j+1} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \tau} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k\};$$

$$(b) \quad \varphi \in \wedge^{k+1} \mathbb{F}^{n+1} \text{ is such that } \frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \tau \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-1}}} = 0 \text{ and } \frac{\partial^{k+1} \varphi}{\partial \xi_2 \dots \partial \xi_k \partial \tau} = 0.$$

Then μ does not satisfy property G3).

Proof. Let us first suppose that μ_0 is of the form (5.2). Then, using the same arguments as in the proof of Lemma 5.3, one can show that $[\mu, [x_{i_{k-1}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [x_{k+1}, \mu]]]]]] \neq 0$, since in its expression the summand $\xi_2 \dots \xi_k \xi_{k+2} \tau$ does not cancel out.

Similarly, if μ_0 is of the form (5.3), then one can show that $[\mu, [x_{i_{k-1}}, \dots, [x_{i_{h+1}}, [x_h, \dots, [x_2, [x_1^2, [1, \mu]]]]]] \neq 0$, since in its expression the summand $\xi_2 \dots \xi_{k+1} \tau$ does not cancel out. \square

Theorem 5.7 Let $\mathcal{P} = PO(n, n+1)$. Suppose that $2 \leq k < n-1$ and that $\mu \in \mathcal{P}_k$. Then (\mathcal{P}, μ) is not a good $(k+1)$ -pair.

Proof. Let us fix a set of odd indeterminates $\xi_1, \dots, \xi_n, \xi_{n+1} = \tau$ and the corresponding basis of monomials of $\wedge(\mathbb{F}^{n+1})$. By Corollary 5.2, if $\mu_0 = 0$ or $\frac{\partial \mu_0}{\partial \tau} = 0$, then μ does not satisfy property G2). Hence suppose that $\frac{\partial \mu_0}{\partial \tau} \neq 0$. Then we may assume, up to a linear change of indeterminates, that $\mu_0 = \xi_1 \dots \xi_k \tau + \varphi$ for some $\varphi \in \wedge^{k+1}(\mathbb{F}^{n+1})$ such that $\frac{\partial^{k+1} \varphi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$. Then, either $\frac{\partial \varphi}{\partial \tau} = 0$ or $\frac{\partial \varphi}{\partial \tau} \neq 0$.

Suppose first $\frac{\partial \varphi}{\partial \tau} = 0$. Then, either for every $r, s > k$ the indeterminates ξ_r, ξ_s appear in different monomials in the expression of φ , or there exist some $r, s > k$ such that ξ_r, ξ_s appear in the same monomial.

In the first case $\mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_k (\xi_{k+1} + \xi_{k+2}) + \rho$ for some $\rho \in \wedge^{k+1}(\mathbb{F}^{n+1})$ such that $\frac{\partial^{k+1} \rho}{\partial \xi_1 \dots \partial \xi_k \partial \xi_{k+1}} = 0 = \frac{\partial^{k+1} \rho}{\partial \xi_1 \dots \partial \xi_k \partial \xi_{k+2}}$. By Corollary 5.2 such an element does not satisfy property G2). Therefore we may assume that there exist some $r, s > k$ such that ξ_r, ξ_s appear in the same monomial, i.e., that, up to a linear change of indeterminates, μ_0 is of the following form:

$$\mu_0 = \xi_1 \dots \xi_k \tau + \xi_1 \dots \xi_h \xi_{k+1} \xi_{k+2} \xi_{i_{h+1}} \dots \xi_{i_{k-1}} + \varphi'$$

for some $\varphi' \in \wedge^{k+1}(\mathbb{F}^{n+1})$ such that $\frac{\partial^{k+1} \varphi'}{\partial \xi_1 \dots \partial \xi_h \partial \xi_{k+1} \partial \xi_{k+2} \partial \xi_{i_{h+1}} \dots \partial \xi_{i_{k-1}}} = 0$ and $\frac{\partial^{k+1} \varphi'}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$, where $h = \max\{0 \leq j \leq k \mid \frac{\partial^{j+2} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \xi_r \partial \xi_s} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k, \text{ and } r, s > k\}$. Therefore μ satisfies hypothesis 1. of Lemma 5.6, hence it does not satisfy property G3).

Now suppose $\frac{\partial \varphi}{\partial \tau} \neq 0$. Then

$$\mu_0 = \xi_1 \dots \xi_k \tau + \xi_{i_1} \dots \xi_{i_h} \tau \xi_{i_{h+1}} \dots \xi_{i_k} + \psi$$

for some $i_1 < \dots < i_h \leq k < i_{h+1} < \dots < i_k$, for some $\psi \in \wedge^{k+1}(\mathbb{F}^{n+1})$ such that $\frac{\partial^{k+1} \psi}{\partial \xi_{i_1} \dots \partial \xi_{i_k} \partial \tau} = 0$ and $\frac{\partial^{k+1} \psi}{\partial \xi_1 \dots \partial \xi_k \partial \tau} = 0$, where $h = \max\{0 \leq j < k \mid \frac{\partial^{j+1} \mu_0}{\partial \xi_{i_1} \dots \partial \xi_{i_j} \partial \tau} \neq 0, \text{ for some } i_1 < \dots < i_j \leq k\}$. Now, up to a permutation of indices, we may assume that $\{i_1, \dots, i_h\} = \{1, \dots, h\}$ and $i_{h+1} = k+1$. Then, either μ does not satisfy property G2), or we may also assume that $\frac{\partial^k \psi}{\partial \xi_2 \dots \partial \xi_k \partial \tau} = 0$. Therefore μ satisfies hypothesis 2. of Lemma 5.6, hence it does not satisfy property G3). \square

Theorem 5.8 Let $\mathcal{P} = PO(n, n+1)$. If (\mathcal{P}, μ) is a good $(k+1)$ -pair, then, up to isomorphisms, one of the following possibilities occur:

a) If $n = 2h + 1$:

a1) $k = 1$ and $\mu_0 = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$;

a2) $k = n$ and $\mu_0 = \xi_1 \dots \xi_{n+1}$.

b) If $n = 2h$:

b1) $k = n$ and $\mu_0 = \xi_1 \dots \xi_{n+1}$.

Proof. By Theorem 5.7, the only possibilities for k are $k = 1$, $k = n - 1$ or $k = n$.

By Corollary 5.2, $\frac{\partial \mu_0}{\partial \xi_i} \neq 0$ for every $i = 1, \dots, n + 1$. It follows that, due to the classification of non-degenerate skew-symmetric bilinear forms, the case $k = 2$ can occur only if $n = 2h + 1$ and, up to equivalence, $\mu_0 = \sum_{i=1}^{h+1} \xi_i \xi_{i+h+1}$, hence we get a1).

If $k = n$ then, up to rescaling the odd indeterminates, $\mu_0 = \xi_1 \dots \xi_n \xi_{n+1}$ and we get cases a2) and b1).

Now assume $k = n - 1$. Then, using the same argument as in the proof of Theorem 5.5, one can show that, up to a linear change of indeterminates, we may assume $\mu_0 = \xi_1 \dots \xi_{n-1} \xi_{n+1} + f$ for some $f \in \wedge^n(\mathbb{F}^{n+1})$ such that $\frac{\partial f}{\partial \xi_{n+1}} = 0$. If $f = 0$ then μ does not satisfy property G2) by Corollary 5.2. If $f \neq 0$, then, up to a linear change of indeterminates, $\mu_0 = \xi_1 \dots \xi_{n-1} \xi_{n+1} + \xi_1 \dots \xi_n = \xi_1 \dots \xi_{n-1}(\xi_{n+1} + \xi_n)$. Then, by Proposition 5.1, μ does not satisfy property G2). \square

6 The classification theorem

Remark 6.1 For every invertible element $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$, the following change of indeterminates preserves the odd symplectic form, i.e., the bracket in $HO(n, n)$, and maps $\varphi \xi_1 \dots \xi_n$ to $\xi'_1 \dots \xi'_n$:

$$x'_1 = \int_0^{x_1} \varphi^{-1}(t, x_2, \dots, x_n) dt =: \Phi, \quad \xi'_1 = \varphi \xi_1,$$

$$x'_i = x_i \quad \forall i \neq 1, \quad \xi'_i = \xi_i - \varphi \frac{\partial \Phi}{\partial x_i} \xi_1 \quad \forall i \neq 1.$$

Indeed one can check that $\{x'_i, x'_j\}_{HO} = 0 = \{\xi'_i, \xi'_j\}_{HO}$ and $\{x'_i, \xi'_j\}_{HO} = \delta_{ij}$ for every $i, j = 1, \dots, n$.

Note that the same change of variables, with the extra condition $\tau' = \tau$, preserves the bracket in the Lie superalgebra $KO(n, n + 1)$, and maps $\varphi \xi_1 \dots \xi_n \tau$ to $\xi'_1 \dots \xi'_n \tau'$.

Theorem 6.2 *A complete list, up to isomorphisms, of good k -pairs with $k > 2$, is the following:*

- i) $(\mathcal{P}^\varphi, \varphi^{-1} \mu)$ with $\mathcal{P} = PO(n, n)$, $n > 2$, $k = n$, $\mu = \xi_1 \dots \xi_n$, $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$;
- ii) $(\mathcal{P}^\varphi, \varphi^{-1} \mu)$ with $\mathcal{P} = PO(n, n + 1)$, $n > 1$, $k = n + 1$, $\mu = \xi_1 \dots \xi_n \tau$, $\varphi \in \mathbb{F}[[x_1, \dots, x_n]]$.

Proof. Let $\mathcal{P} = PO(n, n)$ with the grading of type $(0, \dots, 0 | 1, \dots, 1)$, and let (\mathcal{P}, μ) be a good k -pair for $k > 2$. Then, by Theorem 5.5, we have necessarily $n > 2$, $k = n$, and $\mu_0 = \xi_1 \dots \xi_n$. It follows that $\mu = \xi_1 \dots \xi_n \psi$ for some invertible element ψ in $\mathbb{F}[[x_1, \dots, x_n]]$. By Remark 6.1, up to a change of variables, we may assume $\psi = 1$. In Example 4.5 we showed that the pair $(\mathcal{P}, \xi_1 \dots \xi_n)$ is a good n -pair. Statement i) then follows from Theorem 2.5, Remark 2.9 and Remark 4.8.

Likewise, if $\mathcal{P} = PO(n, n + 1)$ with the grading of type $(0, \dots, 0 | 1, \dots, 1, 1)$ and (\mathcal{P}, μ) is a good k -pair for $k > 2$, by Theorem 5.8 we have necessarily $n > 1$, $k = n + 1$ and $\mu_0 = \xi_1 \dots \xi_n \tau$. It follows that $\mu = \xi_1 \dots \xi_n \tau \psi$ for some invertible element ψ in $\mathbb{F}[[x_1, \dots, x_n]]$. Again by Remark 6.1, we may assume $\psi = 1$. Furthermore in Example 4.7 we showed that $(\mathcal{P}, \xi_1 \dots \xi_n \tau)$ is a good n -pair. Statement ii) then follows from Theorem 2.5, Remark 2.9 and Remark 4.8. \square

Theorem 6.3 *Let $n > 2$.*

- a) *Any simple linearly compact generalized n -Nambu-Poisson algebra is gauge equivalent either to the n -Nambu algebra or to the n -Dzhumadil'daev algebra.*
- b) *Any simple linearly compact n -Nambu-Poisson algebra is isomorphic to the n -Nambu algebra.*

Proof. By Theorems 4.12 and 2.5, we first need to consider good n -pairs $(\mathcal{P}^\varphi, \mu)$ where $\mathcal{P} = PO(k, k)$ or $\mathcal{P} = PO(k, k+1)$ and $n > 2$. A complete list, up to isomorphisms, of such pairs is given in Theorem 6.2. The statement then follows from the construction described in Proposition 4.10. We point out that the pair $(\mathcal{P}^\varphi, \varphi^{-1}\xi_1 \dots \xi_n)$, with $\mathcal{P} = PO(n, n)$, corresponds to \mathcal{N}^φ where \mathcal{N} is the n -Nambu algebra; similarly, the pair $(\mathcal{P}^\varphi, \varphi^{-1}\xi_1 \dots \xi_n \tau)$, with $\mathcal{P} = PO(n, n+1)$, corresponds to \mathcal{N}^φ , where \mathcal{N} is the n -Dzhumadildaev algebra (see also Remark 4.13). \square

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